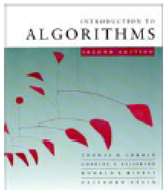
Universal Hashing

A weakness of hashing

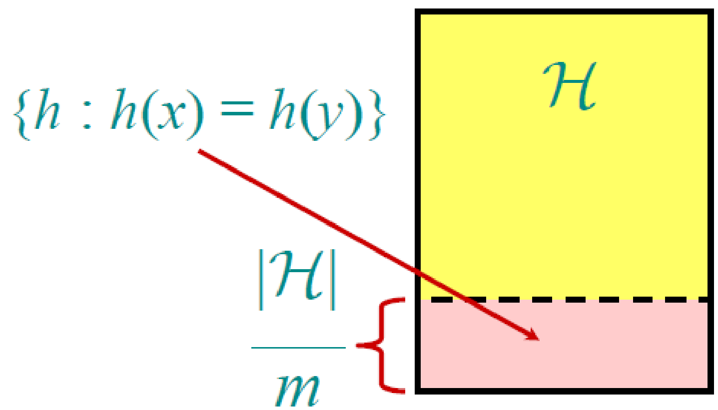
- Problem
  - For any hash function h, a set of keys exist that can cause the average access time of a hash table to skyrocket.
    - An adversary can pick all the keys which all map to the same bucket
- Idea
  - Choose the hash function at random, independent of the keys



Universal hashing

Definition. Let U be a universe of keys, and let $\mathcal{H}$ be a finite collection of hash functions, each mapping U to $\{0, 1, \ldots, m-1\}$. We say $\mathcal{H}$ is ***universal*** if for all $x, y \in U$, where $x \neq y$, we have $|\{h \in \mathcal{H} : h(x) = h(y)\}| = |\mathcal{H}|/m$.

That is, the chance of a collision between x and y is $1/m$ if we choose h randomly from $\mathcal{H}$.



Universality is good

Theorem. Let h be a hash function chosen (uniformly) at random from a universal set $\mathcal{H}$ of hash functions. Suppose h is used to hash n arbitrary keys into the m slots of a table T. Then, for a given key x, we have

$$E[\#\text{collisions with } x] < n/m.$$

Proof of theorem

Proof. Let C_x be the random variable denoting the total number of collisions of keys in T with x, and let

$$c_{xy} = \begin{cases} 1 & \text{if } h(x) = h(y), \\ 0 & \text{otherwise.} \end{cases}$$

Note: $E[c_{xy}] = 1/m$ and $C_x = \sum_{y \in T - \{x\}} c_{xy}$.

Proof (continued)

$$E[C_x] = E\left[\sum_{y \in T - \{x\}} c_{xy}\right]$$

- Take expectation of both sides.

$$= \sum_{y \in T - \{x\}} E[c_{xy}]$$

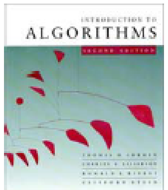
- Linearity of expectation.

$$= \sum_{y \in T - \{x\}} 1/m$$

- $E[c_{xy}] = 1/m$.

$$= \frac{n-1}{m} . \quad \square$$

- Algebra.



Constructing a set of universal hash functions

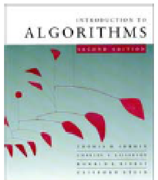
Let m be prime. Decompose key k into $r + 1$ digits, each with value in the set $\{0, 1, \ldots, m-1\}$. That is, let $k = \langle k_0, k_1, \ldots, k_r \rangle$, where $0 \le k_i < m$.

Randomized strategy:

Pick $a = \langle a_0, a_1, \ldots, a_r \rangle$ where each a_i is chosen randomly from $\{0, 1, \ldots, m-1\}$.

Define $h_a(k) = \sum_{i=0}^{r} a_i k_i \bmod m$. ***Dot product, modulo m***

How big is $\mathcal{H} = \{h_a\}$? $|\mathcal{H}| = m^{r+1}$. ← **REMEMBER THIS!**



Universality of dot-product hash functions

Theorem. The set $\mathcal{H} = \{h_a\}$ is universal.

Proof. Suppose that $x = \langle x_0, x_1, \ldots, x_r \rangle$ and $y = \langle y_0, y_1, \ldots, y_r \rangle$ be distinct keys. Thus, they differ in at least one digit position, wlog position 0. For how many $h_a \in \mathcal{H}$ do x and y collide?

We must have $h_a(x) = h_a(y)$, which implies that

$$\sum_{i=0}^{r} a_i x_i \equiv \sum_{i=0}^{r} a_i y_i \pmod{m} .$$

Proof (continued)

Equivalently, we have

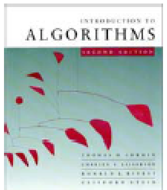
$$\sum_{i=0}^{r} a_i(x_i - y_i) \equiv 0 \pmod{m}$$

or

$$a_0(x_0 - y_0) + \sum_{i=1}^{r} a_i(x_i - y_i) \equiv 0 \pmod{m} \ ,$$

which implies that

$$a_0(x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i(x_i - y_i) \pmod{m} \ .$$



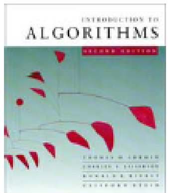
Fact from number theory

Theorem. Let m be prime. For any $z \in \mathbb{Z}_m$ such that $z \neq 0$, there exists a unique $z^{-1} \in \mathbb{Z}_m$ such that

$$z \cdot z^{-1} \equiv 1 \pmod{m}.$$

Example: $m = 7$.

z	1	2	3	4	5	6
z^{-1}	1	4	5	2	3	6



Back to the proof

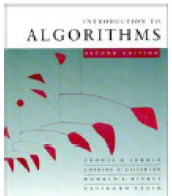
We have

$$a_0(x_0 - y_0) \equiv -\sum_{i=1}^{r} a_i(x_i - y_i) \pmod m,$$

and since $x_0 \neq y_0$, an inverse $(x_0 - y_0)^{-1}$ must exist, which implies that

$$a_0 \equiv \left(-\sum_{i=1}^{r} a_i(x_i - y_i)\right) \cdot (x_0 - y_0)^{-1} \pmod m.$$

Thus, for any choices of $a_1, a_2, \ldots, a_r$, exactly one choice of a_0 causes x and y to collide.



Proof (completed)

Q. How many h_a's cause x and y to collide?

A. There are m choices for each of $a_1, a_2, \ldots, a_r$, but once these are chosen, exactly one choice for a_0 causes x and y to collide, namely

$$a_0 = \left(\left(-\sum_{i=1}^{r} a_i (x_i - y_i) \right) \cdot (x_0 - y_0)^{-1} \right) \bmod m .$$

Thus, the number of h 's that cause x and y to collide is $m^r \cdot 1 = m^r = |\mathcal{H}|/m$. ∎